

# Research Presentation

The circle method and Waring's problem

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# Waring's problem

Lagrange, 1770

Every positive integer is a sum of four squares.

Waring's Conjecture, 1770

All natural numbers are the sum of at most 4 squares, 9 cubes, or 19 fourth powers.

In the case of squares:

- ▶  $23 = 9 + 9 + 4 + 1 = 3^2 + 3^2 + 2^2 + 1^2$
- ▶  $50 = 49 + 1 = 7^2 + 1^2$

In the case of cubes:

- ▶  $23 = 8 + 8 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 2^3 + 2^3 + 7 \cdot 1^3$
- ▶  $100 = 64 + 27 + 8 + 1 = 4^3 + 3^3 + 2^3 + 1^3$

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Modern Problem

For  $k \geq 2$ , what is the value of the function  $G(k)$ , the smallest number  $s$  such that every sufficiently large number is the sum of at most  $s$   $k$ -th powers of positive integers.

We are interested in the representation function

$$R(N) := \# \left\{ (x_1, \dots, x_s) \in \mathbb{N}^s : 1 \leq x_i \leq P, x_1^k + \dots + x_s^k = N \right\},$$

where  $P \approx N^{1/k}$ .

## Some early results...

- ▶ Hilbert provided an *affirmative* answer in 1909: For each  $k$ , there exists some finite number  $s$  such that every integer is the sum of at most  $s$   $k$ -th powers.

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- ▶ In the 1920s, Hardy and Littlewood's *Partitio Numerorum* papers produced asymptotic formulas for the representation function  $R(N)$ .
- ▶ In 1930s, Vinogradov later reformulated the method using finite exponential sums, making the analysis more direct.
- ▶ In 1938, Hua Luogeng used his inequality to better control the exponential sums on the minor arcs and obtained the classical bound  $G(k) \leq 2^k + 1$ .

## Recent developments

More recently, work by Vaughan, Wooley, and others has further refined the circle method, leading to increasingly tight bounds for  $G(k)$ .

Wooley's development of *efficient congruencing* in the 1990s and 2000s brought the general bound for large  $k$  down to essentially  $G(k) \ll k(\log k + \log \log k)$ .

Additionally, Bourgain–Demeter–Guth's *decoupling method* has introduced new perspectives on bounding exponential sums.

## How the circle method works?

Fix a large integer  $N$  and set  $P \approx N^{1/k}$ . Define

$$T(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^k) \quad \text{and} \quad R(N) = \int_0^1 T(\alpha)^s e(-N\alpha) d\alpha.$$

- ▶ Expanding  $T(\alpha)^s$  lists all  $s$ -tuples  $(x_1, \dots, x_s)$ .
- ▶ The integral keeps exactly the tuples with  $x_1^k + \dots + x_s^k = N$ .
- ▶ Split the integral as  $\int_{\mathfrak{M}} + \int_{\mathfrak{m}}$ , where  $[0, 1] = \mathfrak{M} \cup \mathfrak{m}$ .

### Major arcs $\mathfrak{M}$

- ▶ Easier.
- ▶ Main term.

### Minor arcs $\mathfrak{m}$

- ▶ Harder!
- ▶ Error term.

## Demo: The circle method picture

In more concrete terms, for a small parameter  $\delta > 0$ , the major arc around  $a/q$  is

$$\mathfrak{M}_{a,q} = \left\{ \alpha \in [0, 1] : \left| \alpha - \frac{a}{q} \right| \leq P^{-k+\delta} \right\}.$$

Then

$$\mathfrak{M} = \bigcup_{1 \leq q \leq P^\delta} \bigcup_{\substack{a \bmod q \\ \gcd(a,q)=1}} \mathfrak{M}_{a,q}, \quad \mathfrak{m} = [0, 1] \setminus \mathfrak{M}.$$



- ▶ The intervals are drawn much wider than their true width  $P^{-k+\delta}$ .
- ▶ With  $\delta < 1/3$ , these major arcs are disjoint.
- ▶ The major arcs contribute to the main term; the minor arcs go into the error term.

## Setup

Recall we are interested in the following representation function:

$$R(N) := \#\{(x_1, \dots, x_s) \in \mathbb{N}^s : 1 \leq x_i \leq P, x_1^k + \dots + x_s^k = N\}.$$

The igniting spark in the circle method is the following *character orthogonality*:

$$\int_0^1 e(n\alpha) \, d\alpha = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \in \mathbb{Z} \setminus \{0\}. \end{cases} \quad (1)$$

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This allows us to write

$$R(N) = \int_0^1 T(\alpha)^s e(-N\alpha) \, d\alpha, \quad (2)$$

where

$$e(x) = \exp(2\pi i x), \quad T(\alpha) = \sum_{1 \leq x \leq P} e(\alpha x^k), \quad \text{and} \quad P = \lceil N^{1/k} \rceil.$$

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Slogan

Counting becomes integration.

## If minor arcs are hard, how do we deal with them?

Orthogonality allows us to turn  $R(N)$  into the two integrals:

$$R(N) = \int_{\mathfrak{M}} T(\alpha)^s e(-N\alpha) d\alpha + \int_{\mathfrak{m}} T(\alpha)^s e(-N\alpha) d\alpha.$$

- ▶ On  $\mathfrak{M}$ ,  $\alpha$  is near a simple fraction  $a/q$ , so  $T(\alpha)$  has arithmetic structure.
- ▶ On  $\mathfrak{m}$ ,  $\alpha$  is away from those fractions. The trivial bound  $|T(\alpha)| \leq P$  is too weak.

### Goal

Show that  $T(\alpha)$  is small enough on  $\mathfrak{m}$ , so the contribution from minor arcs forms the error term.

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### Answer

We need **Weyl** inequality for *point-wise cancellation* and **Hua** for *high-moment cancellation*.

## For the sake of completeness

### Lemma (Weyl's inequality)

Let  $k \geq 2$  and  $f(x) = \alpha x^k + \alpha_1 x^{k-1} + \dots + \alpha_k$  be a degree  $k$  real polynomial with leading coefficient  $\alpha$ . Assume that  $\alpha$  has a rational approximation  $a/q$ , satisfying

$$\gcd(a, q) = 1, \quad q > 0, \quad \left| \alpha - \frac{a}{q} \right| \leq \frac{1}{q^2}.$$

Then, for all  $\varepsilon > 0$ ,

$$\left| \sum_{x \leq P} e(f(x)) \right| \ll_{k, \varepsilon} P^{1+\varepsilon} \left( \frac{1}{P} + \frac{1}{q} + \frac{q}{P^k} \right)^{1/2^{k-1}}.$$

It gives cancellation for exponential sums away from good rational approximations, so it controls the minor arcs point-wise.

## For the sake of completeness

### Lemma (Hua's inequality)

*If*

$$T(\alpha) := \sum_{x=1}^P e(\alpha x^k),$$

*then for any fixed  $\varepsilon > 0$ , we have*

$$\int_0^1 |T(\alpha)|^{2^k} d\alpha \ll P^{2^k - k + \varepsilon}.$$

It gives a strong average bound for high powers of the same exponential sum.

## Contribution from the minor arcs

The trivial bound is too weak:

$$|T(\alpha)| \leq P.$$

On the minor arcs, rational approximations have no small denominator.

So when we combine the power saving from *Weyl's inequality* and mean-value control from *Hua's inequality*, we will have

### Lemma

If  $s \geq 2^k + 1$ , then we have

$$\int_{\mathfrak{m}} |T(\alpha)|^s d\alpha \ll P^{s-k-\delta'},$$

where  $\delta'$  is a positive number dependent on  $\delta$ .

## Major arcs I: Approximating $T(\alpha)$

Let  $\alpha \in \mathfrak{M}_{a,q}$  and  $\beta = \alpha - \frac{a}{q}$ .

We define

$$S_{a,q} = \sum_{1 \leq x \leq q} e\left(\frac{ax^k}{q}\right), \quad I(\beta) = \int_0^P e(\beta t^k) dt.$$

Then we can obtain

$$T(\alpha) = q^{-1} S_{a,q} I(\beta) + O(P^{2\delta}).$$

On one major arc, the approximation gives

### Major-arc approximation

$$\int_{\mathfrak{M}_{a,q}} T(\alpha)^s e(-N\alpha) d\alpha \approx (q^{-1} S_{a,q})^s e\left(-N\frac{a}{q}\right) \int_{|\beta| < P^{-k+\delta}} I(\beta)^s e(-N\beta) d\beta.$$

## Major arcs II: Getting the main term

Now we sum over all admissible  $a, q$ , we obtain the contribution from the major arcs:

### Lemma

Recall that  $\mathfrak{M}$  denotes the totality of the major arcs  $\mathfrak{M}_{a,q}$ . We have

$$\int_{\mathfrak{M}} T(\alpha)^s e(-N\alpha) d\alpha = P^{s-k} \mathfrak{S}(P^\delta, N) J(P^\delta) + O(P^{s-k-\delta'}),$$

for some  $\delta' > 0$ , where

$$\mathfrak{S}(P^\delta, N) = \sum_{q \leq P^\delta} \sum_{\substack{1 \leq a \leq q \\ \gcd(a,q)=1}} (q^{-1} S_{a,q})^s \cdot e\left(-N \frac{a}{q}\right),$$

$$J(P^\delta) = \int_{|\gamma| < P^\delta} \left( \int_0^1 e(\gamma t^k) dt \right)^s e(-\gamma) d\gamma.$$

## Assembling everything: The final asymptotic formula

We now assemble the two arcs estimates: For some  $\delta' > 0$ ,

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$$R(N) = P^{s-k} \mathfrak{S}(P^\delta, N) J(P^\delta) + O(P^{s-k-\delta'})$$

- ▶ Major arcs give the main term and minor arcs are absorbed into the error term.
- ▶ Absolute convergence lets  $\mathfrak{S}(P^\delta, N) \rightarrow \mathfrak{S}(N)$  and the singular-integral estimate gives  $J(P^\delta) \rightarrow C_{k,s}$ .

Since  $P = \lceil N^{1/k} \rceil$ ,  $P^{s-k}$  becomes  $N^{s/k-1}$  up to some small error. For some  $\rho > 0$ ,

### Asymptotic formula

$$R(N) = C_{k,s} N^{s/k-1} \mathfrak{S}(N) + O\left(N^{s/k-1-\rho}\right), \quad \text{where} \quad C_{k,s} = \frac{\Gamma(1 + 1/k)^s}{\Gamma(s/k)}.$$

## Singular series

The asymptotic formula is only useful if *the main term is positive*:

$$R(N) = C_{k,s} N^{s/k-1} \mathfrak{S}(N) + O(N^{s/k-1-\rho}).$$

- ▶ The singular series can be rewritten as a product of local factors, one for each prime  $p$ .
- ▶ These factors ask whether the congruence

$$x_1^k + \cdots + x_s^k \equiv N \pmod{p^v}$$

has enough solutions as  $v \rightarrow \infty$  (*Hasse's local-global principle*).

- ▶ The key conclusion is that, for  $s \geq 2^k + 1$ ,

$$\mathfrak{S}(N) \geq C_1(k, s) > 0,$$

where  $C_1(k, s)$  is some constant.

# Conclusion

## Main theorem

Every sufficiently large integer can be written as a sum of  $s$  positive integral  $k$ -th powers for  $s \geq 2^k + 1$ .

Now we revisit the steps:

1. Orthogonality turns Waring's problem into an integral of an exponential sum.
2. Major arcs produce the main term.
3. Weyl's and Hua's inequalities make the minor arcs smaller than the main term.
4. The singular series checks the local congruence information and proves that the main term is in fact positive.

Therefore, we now have a self-contained proof of the classical bound

$$G(k) \leq 2^k + 1.$$

Thank you!